ASYMPTOTIC PROPERTIES OF EIGENVALUES IN PROBLEMS OF THE THEORY OF THIN ELASTIC SHELLS

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We shall consider in the following linear problems of free vibrations and stability of thin elastic shells. The ultimate aim is an investigation concerning asymptotic behavior of eigenvalues depending on the density and configuration of the nodal lines of the eigenfunctions.

1. It has been shown [1] that in many cases an approximate determination of the states of stress and strain of a thin elastic shell can be achieved by integration of the system of equations

$$\begin{gathered} L\,(C) - a^2R^2N\,(2EhW) + Z = 0 \\ L\,(2EhW) + N\,(C) = 0 \end{gathered} \qquad \left(a^2 = \frac{h^2}{3R^2\,(1-\sigma^2)}\right) \tag{1.1}$$

where C denotes a stress function in terms of which we can easily express the stress resultants of the shell; W is the normal deflection, while R is a certain characteristic curvature radius; Z is the normal component of the external surface loading; L and N are differential operators defined by

$$L = \frac{1}{AB}\left(\frac{\partial}{\partial\alpha}\frac{B}{A}\frac{1}{R_2}\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\frac{A}{B}\frac{1}{R_1}\frac{\partial}{\partial\beta}\right)$$

$$N = \Delta\Delta, \qquad \Delta = \frac{1}{AB}\left(\frac{\partial}{\partial\alpha}\frac{B}{A}\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\frac{A}{B}\frac{\partial}{\partial\beta}\right)$$

Equations (1.1) are based upon the assumption that the middle surface of the shell is referred to the curvature lines and that the first quadratic form of the surface can be represented by the formula

$$I = A^2d\alpha^2 + B^2d\beta^2$$

R_1 and R_2 are the principal radii of curvature, and it is assumed that the coefficients appearing in L and N are bounded in the region considered.

Starting from (1.1) we can construct approximate expressions for the states of stress and strain of the shell in such cases when their variability is sufficiently large. It will be shown in the following that this is sufficient for the present investigation.

2. In problems on vibrations and static and dynamic stability we have to put

$$Z = -\frac{m}{2Eh}\frac{\partial^2}{\partial t^2}(2EhW) + (q_0 + q_t)\ M\ (2EhW) \qquad (2.1)$$

where m is the mass of the unit area of the shell, while

$$M = T_1\left(\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{A}\frac{\partial}{\partial\alpha} + \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial}{\partial\beta}\right) + T_2\left(\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{B}\frac{\partial}{\partial\beta} + \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial}{\partial\alpha}\right) +$$
$$+ S_1\left(\frac{1}{A}\frac{\partial}{\partial\alpha}\frac{1}{B}\frac{\partial}{\partial\beta} - \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial}{\partial\alpha}\right) - S_2\left(\frac{1}{B}\frac{\partial}{\partial\beta}\frac{1}{A}\frac{\partial}{\partial\alpha} - \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial}{\partial\beta}\right)$$

We assume that:

a) the shell is acted upon by some surface loading whose intensity is determined by the quantity $q_0 + q_1$, where q_0 is a constant and q_t is a variable (with respect to time) component of the loading;

b) sufficiently small values of $q_0 + q_1$ produce in the shell a state of membrane stresses defined by the values of the tangential stress resultants

$$T_1^\circ = 2Eh\,(q_0 + q_t)\,T_1, \qquad T_2^\circ = 2Eh\,(q_0 + q_t)\,T_2$$
$$S_1^\circ = 2Eh\,(q_0 + q_t)\,S_1, \qquad S_2^\circ = 2Eh\,(q_0 + q_t)\,S_2$$

where T_1, T_2, S_1, S_2 are given functions of α, β. We omit here, as is often done, the tangential components of the inertia forces in problems of stability and those of the additional reduced loading in problems of dynamics.

3. Let us apply Galerkin's method to the solution of the system (1.1), (2.1). To this end we substitute

$$C = \gamma c, \qquad 2EhW = \zeta w \qquad (3.1)$$

where c, w are given functions, while γ, ζ are unknown numbers (in problems of static stability) or unknown functions of t (in problems of vibrations and dynamic stability). Multiplying the first of Equations (1.1) by w and the second by c and integrating the equalities obtained, we get equations for the determination of γ and ζ:

$$\gamma\iint L\,(c)\,wAB\,d\alpha\,d\beta - a^2R^2\zeta\iint N\,(w)\,wAB d\alpha d\beta -$$
$$-\frac{m}{2Eh}\frac{d^2\zeta}{dt^2}\iint w^2AB\,d\alpha\,d\beta + (q_0+q_t)\,\zeta\iint M\,(w)wAB\,d\alpha\,d\beta = 0 \qquad (3.2)$$
$$\gamma\iint N\,(c)\,cAB\,d\alpha\,d\beta + \zeta\iint L\,(w)\,cAB\,d\alpha\,d\beta = 0$$

4. Substituting $q_0 + q_t = 0$ in (3.2) and replacing γ, ζ by $\gamma\cos\omega t$ and $\zeta\cos\omega t$, respectively, we find for the frequency of free vibrations of the shell the formula

$$\frac{m}{2Eh}\,\omega^2 = k^{2\rho-4}\,\frac{1}{\nu'}\left(l + k^{8-2\rho-2/\tau]}\,\frac{R^2}{3\,(1-\sigma^2)}\,n\right) \qquad (4.1)$$

where k is, for the case of a thin shell, a large parameter:

$$k = \left(\frac{h}{R}\right)^{-\tau} \qquad (\tau > 0) \qquad (4.2)$$

(τ is an arbitrary positive number), while l, n and ν' are numbers defined by the formulas

$$k^{2\rho}l = \iint L\,(c)\,wABd\alpha\,d\beta\iint L\,(w)\,cABd\alpha\,d\beta$$
$$k^8n = \iint N\,(w)\,wAB\,d\alpha\,d\beta\iint N\,(c)\,cAB\,d\alpha\,d\beta \qquad (4.3)$$
$$k^4\nu' = \iint N\,(c)\,cAB\,d\alpha\,d\beta\iint w^2AB\,d\alpha\,d\beta$$

We use, on the left-hand sides of (4.3), various powers of the parameter k as factors for the purpose of convenience in the subsequent presentation; ρ is, for the time being, an arbitrary number.

Analogously, substituting $q_t = 0$ in (3.2) and assuming that γ, ζ are independent of t, we find for the critical value q_0 in the problem of static stability the formula

$$q_0 = k^{2\rho-\chi-4}\,\frac{1}{\nu}\left(l + k^{8-2\rho-2/\tau}\,\frac{R^2}{3\,(1-\sigma^2)}\,n\right) \qquad (4.4)$$

where, in addition to (4.3), we use the notation

$$k^{4+\chi}\nu = \iint N\,(c)\,cAB\,d\alpha\,d\beta\iint M\,(w)\,wAB\,d\alpha\,d\beta \qquad (4.5)$$

while χ is, for the time being, an arbitrary number.

We can derive without difficulty the differential equation of dynamic stability as well. Following Bolotin [2], we write it in the form

$$\frac{d^2\zeta}{dt^2} + \omega^{\circ 2}\left(1 - \frac{q_0 + q_t}{q_0^{\circ}}\right)\zeta = 0$$

where ω° and q_0° are the values of the numbers ω and q_0 determined by Formulas (4.1) and (4.4). In the following we confine ourselves to an analysis of Formulas (4.1) and (4.4), although the method of investigation is applicable to a study of the properties of the equation of dynamic stability as well.

5. Starting from Formulas (3.1), we write

$$\begin{aligned} c &= c_* \, [r_- \cos k\,(f_1 - f_2) - r_+ \cos k\,(f_1 + f_2)] \\ w &= w_* \, [\cos k\,(f_1 - f_2) - \cos k\,(f_1 + f_2)] \end{aligned} \tag{5.1}$$

where c_*, w_*, f_1, f_2, r_-, r_+ are functions of a, β which are left to our choice, while k is the large parameter (4.2). Prescribing for w_* and c_* the conditions

$$w_* \geqslant 0, \qquad c_* \geqslant 0 \tag{5.2}$$

we consider the properties of the states of stress and strain D, determined by Formulas (3.1), (5.1). Taking (5.2) into account and noting that

$$\cos k(f_1 - f_2) - \cos k(f_1 + f_2) = 2 \sin kf_1 \sin kf_2$$

we conclude that the nodal lines of D (lines along which $w = 0$) can be represented only by level lines of the functions f_1 and f_2. Thus, having chosen f_1 and f_2 in an appropriate manner, the result can be achieved that D will have two systems of nodal lines, each belonging to some family of curves prescribed in advance. These two families can be also reduced to one. To this end it is necessary, for example*, to assume $f_2 = \text{const} \neq (n/\pi)k$. The density of the nodal lines of D will increase with increasing k, i.e. with increasing r at a given h/R. We shall call

* The number of the families of nodal lines can be reduced to one in another way also, namely, starting from the assumption that the level lines of the functions f_1 and f_2 coincide, i.e. that there is a mutual functional dependence between f_1 and f_2. In the interest of definiteness we shall use the method given in the text. This involves, in particular, exclusion of the possibility that the equalities

$$f_1 + f_2 = \text{const}, \quad f_1 - f_2 = \text{const}$$

may be fulfilled at all points of the region considered.

the number r, as in problems of static equilibrium of shells [1], the index of variability.

So we see that a proper choice of f_1 and f_2 permits us to obtain a state of stress and strain (3.1), (5.1) having two (or one) systems of nodal lines belonging to two (or one) families of curves prescribed in advance; a proper choice of the index r of variability permits at fixed f_1 and f_2, the density of the nodal lines to be intensified or reduced; finally, a suitable choice of c_*, w_* permits the boundary conditions of the problem (which are supposed to be homogeneous) to be satisfied, provided that the conditions (5.2) remain valid and that the nodal lines are not changed.

With the aid of (3.1), (5.1) we can obtain a net of nodal lines of any desired density without requiring that f_1, f_2, c_*, w_* have a large variability. Therefore we assume that the functions f_1, f_2, w_*, c_*, r_-, r_+ can be chosen in such a way that their variability does not become too large and that c and w approximate sufficiently, in the sense of proximity of the integrals (4.3), (4.5), some solution of Equations (1.1), (2.1). At such a choice of f_1, f_2, w_*, c_*, r_-, r_+, Formulas (4.1) and (4.4) will yield values of ω^2 and q_0 sufficiently near to the exact values. The assumption introduced will be justified if the following two assumptions are justified:

a) w can be approximated, in the way just indicated, by means of the second of the formulas (5.1) after a suitable choice of f_1, f_2, w_*;

b) r_-, r_+ and c_* can be selected in such a way that at a chosen w the first of the formulas (5.1) gives a sufficient approximation of c.

The assumption (a) can be regarded as justified by the considerations presented in this section. The assumption (b) will be discussed below.

6. The problem now consists in the derivation of asymptotic (for $k \to \infty$) expressions for the integrals (4.3), (4.5) under the assumption that c and w are of the form (5.1). It is easy to ascertain the validity of the formula

$$\begin{aligned} P(a) = &-k^2 \ [P_0^- r_- a_* \cos k(f_1 - f_2) - P_0^+ r_+ a_* \cos k(f_1 + f_2)] - \\ &- k\,[P_1^-(r_- a_*) \sin k(f_1 - f_2) - P_1^+(r_+ a_*) \sin k(f_1 + f_2)] + \\ &+ [P_2(r_- a_*) \cos k(f_1 - f_2) - P_2(r_+ a_*) \cos k(f_1 + f_2)] \end{aligned} \qquad (6.1)$$

where P is an operator of the second order, which may represent L or M; a is a function of the form (5.1), which may represent c or w (if a is identified with w, then each of the quantities r_- and r_+ must be replaced by unity). The other notations in (6.1) are as follows:

$$L_0 = \frac{1}{A^2}\frac{1}{R_2}\left(\frac{\partial f}{\partial \alpha}\right)^2 + \frac{1}{B^2}\frac{1}{R_1}\left(\frac{\partial f}{\partial \beta}\right)^2$$

$$L_1 = \frac{2}{A^2R_2}\frac{\partial f}{\partial \alpha}\frac{\partial}{\partial \alpha} + \frac{2}{B^2R_1}\frac{\partial f}{\partial \beta}\frac{\partial}{\partial \beta} + \frac{1}{A^2R_2}\frac{\partial^2 f}{\partial \alpha^2} + \frac{1}{B^2R_1}\frac{\partial^2 f}{\partial \beta^2} + \tag{6.2}$$

$$+ \frac{1}{AB}\frac{\partial}{\partial \alpha}\left(\frac{B}{AR_2}\right)\frac{\partial f}{\partial \alpha} + \frac{1}{AB}\frac{\partial}{\partial \beta}\left(\frac{A}{BR_1}\right)\frac{\partial f}{\partial \beta}, \qquad L_2 = L$$

$$M_0 = T_1\frac{1}{A^2}\left(\frac{\partial f}{\partial \alpha}\right)^2 + T_2\frac{1}{B^2}\left(\frac{\partial f}{\partial \beta}\right)^2 + (S_1 - S_2)\frac{1}{A}\frac{\partial f}{\partial \alpha}\frac{1}{B}\frac{df}{\partial \beta}, \; M_2 = M$$

(The formula for M_1 is not written down, since it will not be needed);

$$P_j^{\pm} = P_j\,|_{f=f_1\pm f_2} \tag{6.3}$$

The corresponding formula for the operator N is of the form

$$N\,(a) = k^4\,[N_0^- r_- a_* \cos k\,(f_1 - f_2) - N_0^+ r_+ a_* \cos k\,(f_1 + f_2)] + \ldots \tag{6.4}$$

where

$$N_0 = \left[\frac{1}{A^2}\left(\frac{\partial f}{\partial \alpha}\right)^2 + \frac{1}{B^2}\left(\frac{\partial f}{\partial \beta}\right)^2\right]^2, \qquad N_j^{\pm} = N_j\,|_{f=f_1\pm f_2} \tag{6.5}$$

while the points represent terms with factors of the form k^s, where $s < 4$.

7. The following remarks concerning the formulas of Section 6 will be useful in the following.

The expressions $P_j^{\pm}$ and $N_j^{\pm}$ are linear differential operators with respect to the independent variables α, β; their order is indicated by their subscripts. In particular, $P_0^{\pm}$ and $N_0^{\pm}$ are operators of the order zero; they are, in other words, expressions free of symbols of derivatives; therefore, the quantities following $P_0^{\pm}$ and $N_0^{\pm}$ in (6.1) and (6.4) are not enclosed in parentheses.

The coefficients of $P_j^{\pm}$ and $N_j^{\pm}$ depend on the function f, which in turn equals either $f_1 - f_2$, or $f_1 + f_2$, depending on the superscript. Exceptions are $P_2^{\pm}$ and $N_4^{\pm}$ (a case in which the subscript equals the order of the original operator). The function f does not appear in $P_2^{\pm}$ and $N_4^{\pm}$. Therefore the signs (plus) and (minus) are omitted at P_2 in (6.1), since these signs indicate what is meant by f.

The first formula (6.5) shows that N_0 is a positive quantity for any arbitrary surface, no matter what the sign of its Gaussian curvature K may be (here and in the following we disregard the uninteresting case that f_1 and f_2 are constants).

Let us consider the question concerning the sign of the expression L_0, defined by (6.2).

If $K > 0$, then it can be assumed that R_1 and R_2 are positive, and L_0 will be a positive quantity.

If $K = 0$, then, to fix the ideas, we assume $R_1 = \infty$; this leads to

$$L_0 = \frac{1}{A^2}\frac{1}{R_2}\left(\frac{\partial f}{\partial \alpha}\right)^2$$

In general R_2 can change its sign (e.g. in the case of a cylindrical shell, whose cross-section has a point of inflexion); such cases are, however, excluded from consideration here and everywhere in the following. Then we can assume that $R_2 > 0$, and then, in the case of $K = 0$, L_0 will be a non-negative quantity. The first and the third of the formulas (6.2) show that the identity $L_0 \equiv 0$ leads then to the identity $L_1 \equiv 0$.

If $K < 0$, then R_1 and R_2 will be of different sign in the first of Formulas (6.2), and it becomes impossible to make any definite statements regarding the sign of L_0.

8. We now turn to a discussion of the assumption (b) formulated at the end of Section 5. Suppose the preceding assumption (a) is correct and a proper choice of w_*, f_1, f_2 in Formulas (5.1) and (3.1) leads to a sufficient approximation for w. Then we can try to find the stress function c with the aid of the second equation of the system (1.1) and compare the result with the one which can be obtained by a proper selection of c_*, r_-, r_+ in Formulas (5.1) and (3.1). Let us carry out, in an approximate manner, the computations involved, considering the parameter k to be arbitrarily large and keeping in all expressions only the terms with the highest powers of k (in such calculations we shall replace here and in the following the equality sign by the sign $\approx$).

According to (3.1) and (6.4) we have

$$N\,(C) \approx k^4\gamma c_*\,[r_-N_0^-\cos k\,(f_1 - f_2) - r_+N_0^+\cos k\,(f_1 + f_2)] \qquad (8.1)$$

The quantities c_*, r_- and r_+ can be considered to differ from zero. In addition, it has been shown in Section 7 that N_0^+ and N_0^- are positive. Therefore the coefficients of $\cos k(f_1 - f_2)$ and $\cos k(f_1 + f_2)$ on the right-hand side of (8.1) are definitely non-vanishing.

For the principal part of $L(2EhW)$ we obtain, with the aid of (6.1):

in the case that L_0^- and L_0^+ differ from zero:

$$L\,(2EhW) \approx k^2\zeta w_*\,[L_0^-\cos k\,(f_1 - f_2) - L_0^+\cos k\,(f_1 + f_2)] \qquad (8.2)$$

in the case that $L_0^- = L_0^+ = 0$, while L_1^- and L_1^+ differ from zero:

$$L\,(2EhW) \approx -\,k\zeta\,[L_1^-(w_*)\sin k\,(f_1 - f_2) - L_1^+(w_*)\sin k\,(f_1 + f_2)] \quad (8.3)$$

in the case that $L_0^- = L_0^+ = L_1^- = L_1^+ = 0$:

$$L\,(2EhW) \approx \zeta\,[L_2\,(w_*)\cos k\,(f_1 - f_2) - L_2\,(w_*)\cos k\,(f_1 + f_2)] \quad (8.4)$$

It follows from the remarks made in Section 7 that (8.2) always applies to a shell of positive curvature, while (8.2) and (8.4) are possible for a shell of zero curvature; Formula (8.3) can be valid only for a shell of negative curvature.

Replacing in the second of equations (1.1) $N(C)$ by the approximate expression (8.1), and $L(2EhW)$ by the approximate expression (8.2) or (8.4), we can satisfy the obtained equations, after having chosen c_*, r_-, r_+ in that equality in such a manner as to make vanish the coefficients of $\cos k(f_1 - f_2)$ and of $\cos k(f_1 + f_2)$ separately. This shows that the assumption (b) can be considered valid in all cases except, perhaps, the case when the middle surface of the shell has negative curvature.

Note. It is easily seen that if the curvature of the middle surface is positive, and also if the curvature of the middle surface equals zero but the functions f_1 and f_2 are chosen in such a way that L_0^+ and L_0^- differ from zero, the quantities r_- and r_+ will have the same sign.

9. Consider the relations

$$\begin{aligned} &\iint \varphi\,(\alpha, \beta)\cos kf\,(\alpha, \beta)\,d\alpha\,d\beta = k^{-1}H\,(\varphi) \\ &\iint \varphi\,(\alpha, \beta)\sin kf\,(\alpha, \beta)\,d\alpha\,d\beta = k^{-1}H\,(\varphi) \end{aligned} \qquad (f \neq \text{const}) \quad (9.1)$$

where H is a certain (varying from case to case) functional of ϕ bounded as $k \to \infty$. Of course, at $\phi \equiv 0$ also $H(\phi) = 0$, but the latter equality may take place also at $\phi \not\equiv 0$.

We shall prove the relations (9.1) with the simplifying assumption that $f_\alpha' \neq 0$ everywhere in the region. Then, using for the sake of definiteness the first of the relations (9.1), we may write

$$\iint \varphi\,(\alpha, \beta)\cos kf\,(\alpha, \beta)\,d\alpha\,d\beta = \frac{1}{k}\iint \frac{\varphi\,(\alpha, \beta)}{f_\alpha'\,(\alpha, \beta)}\frac{\partial}{\partial\alpha}\sin kf\,(\alpha, \beta)\,d\alpha\,d\beta$$

Integration by parts gives

$$\iint \varphi(\alpha, \beta) \cos kf(\alpha, \beta) = \frac{1}{k} \oint \frac{\varphi(\alpha, \beta)}{f_\alpha'(\alpha, \beta)} \sin kf(\alpha, \beta)\, ds - \\ - \frac{1}{k} \iint \frac{\partial}{\partial\alpha} \left[\frac{\varphi(\alpha, \beta)}{f_\alpha'(\alpha, \beta)} \right] \sin kf(\alpha, \beta)\, d\alpha\, d\beta$$

Both the area integral and the integral around the boundary line of the region are finite (if certain obvious conditions are fulfilled for the functions ϕ and f), and the proof is completed.

From (9.1) we derive the formulas

$$\iint \varphi(\alpha, \beta) \cos kf(\alpha, \beta) \cos kg(\alpha, \beta)\, d\alpha\, d\beta = \tag{9.2}$$

$$= \frac{1}{2} \iint \varphi(\alpha, \beta) \, [\cos k(f - g) + \cos k(f + g)]\, d\alpha\, d\beta =$$

$$= \begin{cases} \dfrac{1}{2} \iint \varphi(\alpha, \beta)\, d\alpha\, d\beta + k^{-1} H(\varphi) \text{ when } f = g \text{ or } f = -g \\ k^{-1} H(\varphi) \text{ when } f - g \neq \text{const and } f + g \neq \text{const} \end{cases}$$

$$\iint \varphi(\alpha, \beta) \cos kf(\alpha, \beta) \sin kg(\alpha, \beta)\, d\alpha\, d\beta = k^{-1} H(\varphi) \tag{9.3}$$

$$\text{when } f - g \neq \text{const}, \quad f + g \neq \text{const}$$

Note. The cases $f - g = \text{const}$ or $f + g = \text{const}$ are of no interest, as implied in the footnote[a] of Section 5.

10. Formulas (9.2) and (9.3) permit the principal parts (at $k \to \infty$) of the integrals appearing in (4.3) and (4.5) to be easily computed.

Taking into consideration the fact that N_0^- and N_0^+ are positive at all points of the region, we obtain

$$\begin{aligned} \iint N(w) wAB\, d\alpha\, d\beta &\approx \frac{k^4}{2} \iint [N_0^- + N_0^+]\, w_*^2 AB\, d\alpha\, d\beta \\ \iint N(c)\, cAB\, d\alpha\, d\beta &\approx \frac{k^4}{2} \iint [r_-^2 N_0^- + r_+^2 N_0^+]\, c_*^2 AB\, d\alpha\, d\beta \\ \iint w^2 AB\, d\alpha\, d\beta &\approx \iint w_*^2 AB\, d\alpha\, d\beta \end{aligned} \tag{10.1}$$

Furthermore, we have the approximate equalities

$$\iint L(c)\, wAB\, d\alpha\, d\beta \approx \iint L(w)\, cAB\, d\alpha\, d\beta \approx$$

$$\approx -\frac{k^2}{2} \iint [L_0^- r_- + L_0^+ r_+]\, c_* w_* AB\, d\alpha\, d\beta \tag{10.2}$$

which are valid only under the condition that

$$\iint [L_0^- r_- + L_0^+ r_+]\, c_* w_* AB\, d\alpha\, d\beta \neq 0 \tag{10.3}$$

If (10.3) is violated in consequence of the condition that at all points of the region under consideration the equalities

$$L_0^- = L_0^+ = 0 \tag{10.4}$$

are fulfilled, then instead of (10.2) we will have

$$\iint L\,(c)\, wAB\, d\alpha\, d\beta = Q_1, \qquad \iint L\,(w)\, cAB\, d\alpha\, d\beta = Q_2 \tag{10.5}$$

Here Q_1, Q_2 are quantities which remain finite at $k \to \infty$. In particular, if in addition to (10.4) the equalities

$$L_1^+ = L_1^- = 0 \tag{10.6}$$

are fulfilled, the relations (10.5) assume the form

$$\begin{aligned} \iint L\,(c)\, wAB\, d\alpha\, d\beta &\approx \frac{1}{2}\iint L_2\,(r_- c_* + r_+ c_*)\, w_* AB\, d\alpha\, d\beta \\ \iint L\,(w)\, cAB\, d\alpha\, d\beta &\approx \frac{1}{2}\iint L_2\,(w_*)\,(r_- c_* + r_+ c_*)\, AB\, d\alpha\, d\beta \end{aligned} \tag{10.7}$$

If, however, (10.3) is violated, while L_0^- and L_0^+ are not identically equal to zero, then instead of (10.2) we will, in general, obtain formulas of the form

$$\iint L\,(c)\, wAB\, d\alpha\, d\beta \approx kE_1, \qquad \iint L\,(w)\, cAB\, d\alpha\, d\beta = kE_2 \tag{10.8}$$

where E_1 and E_2 are quantities which remain finite when $k \to \infty$.

Comparing (10.1), (10.2), (10.5) and (10.8) with Formulas (4.3), we can conclude that in (4.3) we may consider l, n and ν' as representing quantities which remain finite at $k \to \infty$, provided that ρ is chosen correspondingly. Specifically, we must have $\rho = 2$ if the condition (10.3) is fulfilled, and $\rho = 0$ if Equations (10.4) are valid in the entire region considered. If the relations (10.3) and (10.4) are invalid simultaneously, then, in general, we will have to put $\rho = 1$; cases, however, are possible when $\rho = 0$.

In the same way we can verify that in (4.5) we have to put $\chi = 2$ if

$$\iint [M_0^- + M_0^+]\, w^2 AB\, d\alpha\, d\beta \neq 0 \tag{10.9}$$

and $\chi = 0$ if everywhere in the region considered

$$M_0^- = M_0^+ = 0 \tag{10.10}$$

and finally $\chi = 1$ or $\chi = 0$ if both (10.9) and (10.10) are invalid.

11. Let us consider the conditions under which the relations (10.3) and (10.4) are fulfilled.

If the curvature of the middle surface of the shell is positive, then L_0^+, L_0^-, r_-, r_+ are positive quantities (see Sections 7 and 8). It has been assumed, further, that c_*, w_* are non-negative and, of course, not identically equal to zero. Thus for a shell of positive curvature the relation (10.3) is always fulfilled.

If the curvature of the middle surface equals zero, then L_0^-, L_0^+ are non-negative quantities (Section 7). It has been shown, further, that the identity $L_0 \equiv 0$ entails the identity $L_1 \equiv 0$ and, finally, that r_- and r_+ have the same sign if L_0^+ and L_0^- differ from zero. This leads to the conclusion that the relation (10.3) can become invalid in the case under consideration only if Equations (10.4) and (10.6) are fulfilled at every point. The equalities (10.4), if expanded, can be written in the form

$$\frac{1}{A^2}\frac{1}{R_2}\left[\frac{\partial f_1}{\partial \alpha} - \frac{\partial f_2}{\partial \alpha}\right]^2 = 0, \qquad \frac{1}{A^2}\frac{1}{R_2}\left[\frac{\partial f_1}{\partial \alpha} + \frac{\partial f_2}{\partial \alpha}\right]^2 = 0$$

These equalities are fulfilled only in the case that

$$\frac{\partial f_1}{\partial \alpha} \equiv \frac{\partial f_2}{\partial \alpha} \equiv 0.$$

Since the possibility of a mutual functional dependence between f_1 and f_2 is excluded, the relations (10.3) will have to be satisfied under the condition $f_1 = f_1(\beta)$, $f_2 = \text{const}$ (or $f_1 = \text{const}$, $f_2 = f_2(\beta)$).

So we see that in the case of a shell of zero curvature the relations (10.3) are not fulfilled then and only then, when the state of stress and strain (3.1), (5.1) has only one system of nodal lines, consisting of the lines $\beta = \text{const}$, i.e. of asymptotic lines (rectilinear generators) of the middle surface. In this case we have to set $\rho = 0$. In all other cases the condition (10.3) is fulfilled and we have to set $\rho = 2$, as in the case of a shell of positive curvature.

If the curvature of the middle surface of the shell is negative, nothing definite can be stated about the sign of L_0 in general, and the question concerning the choice of values for ρ becomes more complicated. Therefore, shells of negative curvature are excluded from consideration in the following. This excludes also the consideration of the cases when $\rho = 1$, since such cases can occur neither for shells of positive curvature nor for shells of zero curvature (if, as assumed, the sign of R_2 does not change in the latter).

12. We now return to Formula (4.1) in order to investigate with its aid the asymptotic behavior of the frequencies of free vibrations of the shell, assuming that the index of variability τ is positive so that the parameter k is arbitrarily large at a sufficiently small value of h/R. Keeping in (4.1) only the principal parts (as $k \to \infty$) we find

$$\begin{aligned}
&\frac{m}{2Eh}\omega^2 \approx k^{2\rho-4}\frac{l}{\nu'} && \text{for } \tau < \tau_0 \\
&\frac{m}{2Eh}\omega^2 \approx k^{2\rho-4}\frac{1}{\nu'}\left(l + \frac{R}{3(1-\sigma^2)}n\right) && \text{for } \tau = \tau_0 \qquad (12.1)\\
&\frac{m}{2Eh}\omega^2 \approx k^{4-2/\tau}\frac{R^2}{3(1-\sigma^2)}\frac{n}{\nu'} && \text{for } \tau > \tau_0
\end{aligned}$$

where τ_0 is a number defined by the formula

$$\tau_0 = \frac{1}{4-\rho} \qquad (12.2)$$

which we shall call the characteristic value of the index of variability τ. We note that τ_0 is determined by ρ, which can assume only three values (2; 1; 0); correspondingly, there are only three characteristic values of the index of variability, namely 1/2; 1/3; 1/4. These values of the index of variability play a particular role also in problems of static equilibrium in the theory of shells [3]. Taking (4.2) into account we may write

$$k^{2\rho-4} = \left(\frac{h}{R}\right)^{(4-2\rho)\tau}, \qquad k^{4-2/\tau} = \left(\frac{h}{R}\right)^{2-4\tau} \qquad (12.3)$$

We conclude from this result that, according to (12.1), ω^2 decreases with increasing τ, or (for $\rho = 2$) ω^2 does not increase as long as τ remains below its characteristic value τ_0; when τ exceeds the value τ_0, ω^2 starts increasing with increasing τ.

Now consider the case $\rho = 2$, which always occurs in a shell of positive curvature (Section 11); in a shell of zero curvature the case indicated occurs when the system of nodal lines includes at least one family of lines non-coinciding with the rectilinear generators.

According to the first formula (12.1), the frequency of free vibrations will remain, in the cases enumerated above, commensurate with $(h/R)^0$ as long as $\tau < \tau_0 = 1/2$, so that within certain limits the increase of τ is not accompanied by essential increase of the frequencies of the free vibrations. Then at $\tau > \tau_0 = 1/2$ the frequencies of the free vibrations essentially increase, according to the third formula (12.1), with the increase of τ according to the law $(H/R)^{2-4\tau}$. The question as to at what value of the index of variability τ the vibration frequency may become a minimum requires in the present case a more detailed study.

Now let $\rho = 0$. This value will be assumed by ρ in the case that a shell of zero curvature undergoes vibrations with one family of nodal lines running along rectilinear generators.

Formulas (12.1) and (12.3) show in this case that as long as $\tau < \tau_0 = 1/4$ the frequencies of free vibrations will be commensurate with $(h/R)^{4\tau}$, i.e. they will decrease with increasing index of variability τ. At $\tau > \tau_0$ the frequencies of the free vibrations are commensurate with $(h/R)^{2-4\tau}$ and they increase with increasing τ. The characteristic value $\tau_0 = 1/4$ is in correspondence with the minimum frequency. The latter is commensurate with $(h/R)^1$, being thus essentially lower (at arbitrarily small values of h/R) than the lowest frequency that can be obtained at vibrations of a shell of zero curvature in the case $\rho = 2$, i.e. in the case that there is at least one family of nodal lines not running along rectilinear generators.

It should be kept in mind that the increase of the index of variability indicates an increase in the number of nodal lines. Consequently, we have before us an inversion typical for problems of the theory of shells: decrease of eigenvalues with increase in the number of nodal lines. This takes place only until a certain limit – until the index of variability reaches its characteristic value, after which the usual course becomes restored: increase of the eigenvalues at increasing number of nodal lines.

13. We now turn to an analysis of Formula (4.4). Keeping on its right-hand side the principal parts (for $k \to \infty$) only, we obtain

$$q_0 \approx k^{2\rho-\chi-4}\frac{l}{\nu} \qquad \text{for } \tau < \tau_0$$

$$q_0 \approx k^{2\rho-\chi-4}\left(l + \frac{R^2}{3(1-\sigma^2)}n\right)\frac{1}{\nu} \qquad \text{for } \tau = \tau_0$$

$$q_0 \approx k^{4-\chi-2/\tau}\frac{R^2}{3(1-\sigma^2)}\frac{n}{\nu} \qquad \text{for } \tau > \tau_0$$

where τ_0, the characteristic index of variability, has the same meaning as in Section 12. The parameter k is determined by (4.2). Hence

$$k^{2\rho-\chi-4} = \left(\frac{h}{R}\right)^{(4+\chi-2\rho)\tau}, \qquad k^{4-\chi-2/\tau} = \left(\frac{h}{R}\right)^{2-(4-\chi)\tau}$$

where ρ and χ can assume the values 0; 1; 2 only. This leads to the conclusion that with τ increasing $k^{2\rho-\chi-4}$ either decreases or (when $\rho = 2$, $\chi = 0$) maintains its values, while $k^{4-\chi-2/\tau}$ always increases with increasing τ. This in turn shows that $(q_0)_{\min}$ will be commensurate with the quantity

$$k^{4-\chi-2/\tau_0} = \left(\frac{h}{R}\right)^{2-(4-\chi)\tau_0} = \left(\frac{h}{R}\right)^{\lambda} \quad \left(\lambda = 2 - \frac{4-\chi}{4-\rho}\right) \tag{13.1}$$

The corresponding value of q_0 will be reached at $r = r_0$ (except in the case $\rho = 2$, $\chi = 0$, which it is not necessary to consider, as will become evident below).

A question of interest is that concerning the minimum value of the critical force in problems of stability; it is therefore necessary to establish the conditions under which the exponent λ in (13.1) reaches its maximum value; its values are given here in the numerical table for $\rho = 2;\ 1;\ 0$, and $\chi = 2;\ 1;\ 0$.

	$\chi=2$	$\chi=1$	$\chi=0$
$\rho=2$	1	$^1/_2$	0
$\rho=1$	$^4/_3$	1	$^2/_3$
$\rho=0$	$^3/_2$	$^5/_4$	1

Consider a shell of positive curvature. We then have $\rho = 2$ independently of the distribution of the nodal lines of the form of loss of stability. It is obvious that the configuration of these lines can always be chosen in such a way as to have the conditions (10.9), (10.10) either fulfilled, or violated, i.e. so as to have for χ the required value. Consequently, the configuration of the nodal lines must be subjected to the requirement that for given T_1, T_2, S_1, S_2 the condition (10.9) be fulfilled, which entails the equality $\chi = 2$, since then λ assumes the maximum value possible at $\rho = 2$, namely the value 1. Hence, in particular, the conclusion that the case $\rho = 2$ and $\chi = 0$ is of no interest.

Now let the curvature of the shell be zero. Then forms of loss of stability will exist at which $\rho = 0$. It will be a loss of stability with one family of nodal lines coinciding with the rectilinear generators. The numerical table shows, however, that for $\rho = 0$ the values of λ are not smaller than 1 and that they become equal to 1 only for $\chi = 0$. At the same time, the equality $\rho = 0$ is realized only at a completely definite configuration of nodal lines on which it is impossible to impose additional conditions, as in the case of a shell of positive curvature. Therefor, two cases are possible for a shell of zero curvature:

Case 1. At a given pre-critical state of stress (T_1, T_2, S_1, S_2) there are such f_1, f_2, corresponding to loss of stability with one family of nodal lines running along rectilinear generators, for which the condition (10.9) is fulfilled.

Case 2. At a given pre-critical state of stress (T_1, T_2, S_1, S_2) with arbitrary f_1, f_2 corresponding to loss of stability with one family of nodal lines running along rectilinear generators, the condition (10.9) is violated and the condition (10.10) is fulfilled.

In Case 1, loss of stability will take place with one family of nodal lines running along rectilinear generators; the critical value q_0 will then be commensurate with $(h/R)^{3/2}$, while the index of variability will be equal to 1/4. This follows from the fact that f_1, f_2 can be chosen in the case under consideration in such a way as to have $\rho = 0$ and $\chi = 2$, for which, according to the table of λ-values, λ assumes its maximum value equal to 3/2. Case 1 is exemplified by the problem of a cylindrical shell acted upon by external pressure. The pre-critical state of stress will be as follows:

$$T_1 = S_1 = S_2 = 0, \qquad T_2 < 0$$

Consequently

$$M_0 = T_2 \frac{1}{B^2}\left(\frac{\partial f}{\partial \beta}\right)^2$$

In order to have the nodal lines coincide with the rectilinear generators, the functions f_1 and f_2 must be chosen, for example, as follows: $f_1 = f_1(\beta)$, $f_2 =$ const. Then

$$M_0^- = T_2 \frac{1}{B^2}\left(\frac{\partial f_1}{\partial \beta}\right)^2, \qquad M_0^+ = T_2 \frac{1}{B^2}\left(\frac{\partial f_1}{\partial \beta}\right)^2$$

and the condition (10.9) is definitely fulfilled.

In Case 2 we select the functions f_1 and f_2 in such a way that the nodal lines run along the rectilinear generators, which leads to $\rho = 0$ and $\chi = 0$. This is in correspondence with $\lambda = 1$ in our table of λ-values. The same λ-value is obtained for $\rho = 2$, i.e. for the case when the nodal lines have arbitrary configuration, except only that which violates the condition (10.9). This means that in Case 2 the coincidence of the nodal lines with the rectilinear generators does not lead to an essential decrease of the critical load. An example for Case 2 is offered by a cylindrical shell under axial compression. In this case we have

$$T_1 < 0, \qquad T_2 = S_1 = S_2 = 0$$

from which

$$M_0 = T_1 \frac{1}{A^2}\left(\frac{\partial f}{\partial \alpha}\right)^2, \qquad M_0^- = T_1 \frac{1}{A^2}\left(\frac{\partial f_1}{\partial \alpha} - \frac{\partial f_2}{\partial \alpha}\right)^2, \qquad M_0^+ = T_1 \frac{1}{A^2}\left(\frac{\partial f_1}{\partial \alpha} + \frac{\partial f_2}{\partial \alpha}\right)^2$$

Selecting f_1 and f_2 in the same manner as in the preceding example, we find that the equality (10.10) is always fulfilled.

14. Examining the table of λ-values once more, one may notice the following possibility: by selecting, in the case of a shell of zero curvature, f_1 and f_2 in such a way that the nodal lines of the form of loss of stability run along the rectilinear generators, we can reach in some cases a dual effect. This will happen when simultaneously with the equations

$$L_0^+ = L_0^- = 0 \tag{14.1}$$

the equations

$$M_0^+ = M_0^- = 0 \tag{14.2}$$

are also fulfilled. In this case we obtain, firstly, $\rho = 0$ instead of $\rho = 2$ by virtue of (14.1), which leads to an increase of λ, and, secondly, $\chi = 0$ instead of $\chi = 2$ by virtue of (14.2), which leads to a decrease of λ.

Loss of stability takes place at maximum λ, and the question arises whether it is not necessary to choose f_1, f_2 in such a way as to have the equalities (14.1) fulfilled not exactly, but with a certain degree of approximation.

As an example we consider the case of a cylindrical shell twisted by shear forces. The pre-critical state will be determined by

$$T_1 = T_2 = 0, \qquad S_1 = -S_2 = S = \text{const} \neq 0$$

from which

$$M_0^- = \frac{2S}{AB}\left(\frac{\partial f_1}{\partial \alpha} - \frac{\partial f_2}{\partial \alpha}\right)\left(\frac{\partial f_1}{\partial \beta} - \frac{\partial f_2}{\partial \beta}\right), \qquad M_0^+ = \frac{2S}{AB}\left(\frac{\partial f_1}{\partial \alpha} + \frac{\partial f_2}{\partial \alpha}\right)\left(\frac{\partial f_1}{\partial \beta} + \frac{\partial f_2}{\partial \beta}\right)$$

Further, we have

$$L_0 = \frac{1}{A^2}\frac{1}{R_2}\left(\frac{\partial f_1}{\partial \alpha} - \frac{\partial f_2}{\partial \alpha}\right)^2, \qquad L_0^+ = \frac{1}{A^2}\frac{1}{R_2}\left(\frac{\partial f_1}{\partial \alpha} + \frac{\partial f_2}{\partial \alpha}\right)^2$$

It is easily seen that by choosing f_2 = const and $f_1 = f_1(\beta)$, i.e. by identifying the nodal lines with the rectilinear generators, we obtain $\rho = 0$, $\chi = 0$, so that λ will be equal to 1. On the other hand, by taking

$$f_2 = \text{const}, \;\; f_1 = f_1{}^1(\beta) + k^{-1} f_1{}^2(\alpha, \beta)$$

i.e. by securing small deviation of the nodal lines from the rectilinear generators, we obtain

$$M_0^- = M_0^+ = k^{-1}\frac{2S}{AB}\frac{\partial f^2}{\partial \alpha}\frac{\partial f_1{}^1}{\partial \beta}, \qquad L_0^- = L_0^+ = k^{-2}\frac{1}{A^2}\frac{1}{R_2}\left(\frac{\partial f_1{}^2}{\partial \alpha}\right)^2$$

from which $\rho = 0$, $\chi = 1$, and the value of λ increases from 1 to 5/4.

15. The present paper achieves an analysis of the asymptotic properties of eigenvalues in problems of the theory of shells, deriving the laws which become valid starting from a certain sufficiently small value of $(h/R)_0$ of the parameter h/R and are the more pronounced the smaller this parameter. The order of smallness of $(h/R)_0$ will vary from problem to problem. It depends on such parameters of the problem as, for example, the length of the shell, the ratio of the radii of curvature, the characteristics of the variability of the coefficients of Equations (1.1) from point to point, etc. The results presented above have then a sense of reality when the order of magnitude of $(h/R)_0$ remains within the limits encountered in practical conditions. This means that the parameters enumerated above must have values not too large and not too small.

In a doctoral dissertation of 1950 ("On the Equilibrium of Thin Elastic Shells at the Post-Critical Stage") by N.A. Alumiae the asymptotic behavior of critical loads, with the influence of some of the aforementioned parameters taken into account, was investigated with the aid of some other methods. Whenever a comparison was possible, the results obtained by Alumiae coincide with those presented above. Also, the study presented here does not claim completeness, inasmuch as cases were excluded from consideration when the coefficients of Equations (1.1) assume infinitely large values in the domain that is of interest to us, e.g. at the apex of a cone, along the circle which on a torus separates the zones of positive and negative curvatures from each other, etc. It should be noted, further, that we have excluded from consideration shells of negative curvature as well as such shells of zero curvature on which the non-vanishing curvature changes its sign. The method of investigation is based upon the assumption that the parameter k is large. Therefore, the condition that the index of variability r be large will be essential. To extend the obtained results to the case $r = 0$ will be possible only by extrapolation. In this sense, the statement, for example, according to which (see Section 12) the frequency of free vibrations of shells of positive curvature remains commensurate with $(h/R)^{\circ}$, as long as $r < r_0 = 1/2$ (with the value $r = 0$ included by extrapolation) is a conditional statement.

In conclusion, we note that the domain of applicability of the original equations (1.1) is also determined by the requirement that the index of variability of the desired states of stress and strain be positive. Therefore, it would not make any sense to replace (1.1) by exact equations, since then even the method of investigation would become inapplicable.

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